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LETTER TO THE EDITOR

One-dimensional asymmetrically coupled maps with defects

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Abstract. In this letter we study chaotic dynamical properties of an asymmetrically coupled one-dimensional chain of maps. We discuss the existence of coherent regions in terms of the presence of defects along the chain. We find out that temporal chaos is instantaneously localized around one single defect and that the tangent vector jumps from one defect to another in an apparently random way. We measure quantitatively the localization properties by defining an entropy-like function in the space of tangent vectors.

Spatio-temporal chaos has been recently studied in many fields, as Bénard convection, optical turbulence, chemical reaction–diffusion systems, and so on [1]. One of the paradigms of spatial chaos is the chaotic evolution of a spatial pattern. Order and chaos are often considered to be opposite notions in nature. One may think of order as a stable and regular behaviour, and of chaos as unstable and erratic behaviour. Then one may be tempted to conclude that spatial order cannot exist in temporally chaotic systems. There are, however, many examples where spatial order coexists with temporal chaos, e.g. in convection, boundary layers and shear flows.

Generally, it is difficult to study this spatio-temporal behaviour from the full equations, e.g. the Navier–Stokes equations in fluids. As a consequence, to gain more insight and to facilitate computation, model systems with a set of coupled dynamical systems, coupled-map lattice, have been introduced. These are a crude, but non-trivial, approximation of extended systems with discrete space and time, but continuous states [1]. The simplest example is given by a set of N continuous variables x_i which evolve in (discrete) time as

$$x_i(n+1) = (1-\alpha_i-\beta_i)\, f(x_i(n)) + \alpha_i f(x_{i-1}(n)) + \beta_i f(x_{i+1}(n)) \, . \tag{1}$$

Usually, the coupling constants are assumed to be independent of the site i and equal, i.e. $\alpha_i = \beta_i = \gamma$, and periodic boundary conditions are taken.

In some physical problems, e.g. shear flow, boundary layers or convection, there is a privileged direction. This can be introduced in the model (1) by taking asymmetric couplings [2, 3]

$$\alpha_i = \gamma_1 \neq \beta_i = \gamma_2 \, . \tag{2}$$

The system (1)–(2) with periodic boundary conditions shows convective instability [4]. In a reference frame moving with a constant velocity v in some band $[v_{\min}, v_{\max}]$ an initial perturbation $\delta x_k(0)$ grows exponentially with a rate given by a co-moving Lyapunov exponent $\lambda(v)$

$$\delta x_{k+vn}(n) \simeq \delta x_k(0)\, e^{\lambda(v)n} \, . \tag{3}$$

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The situation becomes much more intriguing and interesting if one introduces non-periodic boundary conditions. For example, see [2], in which a unidirected lattice map is used to mimic convective turbulence. Recently Aranson *et al* [5] studied the system (1) with the following coupling constants:

$$\begin{aligned} &\alpha_i = \gamma_1 \qquad \beta_i = \gamma_2 \qquad \text{for} \quad i = 2, \ldots, N-1 \\ &\alpha_1 = 0 \qquad \beta_1 = \gamma_2 \qquad \alpha_N = \gamma_1 \qquad \beta_N = 0 \end{aligned} \qquad \gamma_1 > \gamma_2 \tag{4}$$

and $f(x) = R - x^2$ with $R = 1.67$, for which the behaviour of a single map is chaotic.

The system (1), (4) has the stable uniform solution

$$x_i(n) = \widetilde{x}(n) \qquad \widetilde{x}(n+1) = f(\widetilde{x}(n)). \tag{5}$$

Aranson *et al* [5] found that starting from randomly non-uniform initial conditions, after some iterations the $x_i(n)$ become partially synchronized: $x_i(n) \simeq \widetilde{x}(n)$ for $i < l_c$, while for $i > l_c$ the x_i are spatially irregular. The finite coherence length l_c is due to the numerical noise. Indeed, they found that l_c increases logarithmically with the noise level in the numerical calculations.

The synchronized zone appears for small values of the index i because of the structure of the map: numerical noise, like any other disturbances, propagates from left to the right due to the choice of the γ's. Therefore, starting from site $i = 1$ and moving to the right one observes a growing of numerical noise up until the coherent region is completly destroyed.

This situation seems rather pathological, since the main features of a chaotic system should not be destroyed by the presence of a small noise.

In order to gain intuition on this behaviour, we have studied the evolution of the tangent vector of the system (1), (4). The evolution law of the tangent vector $\boldsymbol{z}$ is obtained linearizing equation (1) about the trajectory

$$z_i(n+1) = (1 - \alpha_i - \beta_i)\, g(x_i(n))\, z_i(n) + \alpha_i g(x_{i-1}(n))\, z_{i-1}(n) + \beta_i g(x_{i+1}(n))\, z_{i+1}(n) \tag{6}$$

where $g(y) = \mathrm{d}f(y)/\mathrm{d}y$, and the coupling constants are given by (4).

A numerical analysis reveals that if the tangent vector is initially localized in the irregular zone of the system, i.e. $z_i(0) = 0$ for $i < l_c$, then one has a behaviour similar to the convective instability. The vector translates, while growing, in the direction of increasing i. However, unlike the case of periodic boundary conditions, as soon as the perturbation reaches the boundary $i = N$, its amplitude decreases rapidly to zero.

On the other hand, if $z_i(0) \neq 0$ for some $i < l_c$, the instability does not travel and there is an exponential growth of $\boldsymbol{z}(n)$ in the synchronized part of the chain. This means that the chaotic part of the system is the spatially coherent one.

A measure of the degree of chaos is given by the Lyapunov exponents. They measure the growth of the tangent vector for large time, and hence they are ruled by the synchronized part of the system. As a consequence, since l_c depends on the noise level in the numerical calculations, we expect that also the Lyapunov exponents should be affected. This is confirmed by the numerical analysis. For example, in the case of $N = 70$ we find only one positive Lyapunov exponent, and the others negative, in a four-byte precision calculation, while an eight-byte calculation leads to several positive exponents.

As stated above, a similar scenario is rather pathological in the framework of chaotic systems, since one expects a sort of structural stability for the main features. In our opinion the origin of this odd behaviour is in the open boundary conditions. Indeed, for a one-dimensional system, open boundary condition represents a very strong 'defect'. The chain is broken somewhere. For this reason we have studied a slightly modified version of the model which in some sense interpolate between the periodic and the open boundary conditions.

We consider the system (1), (2), with $\gamma_1 > \gamma_2$ and periodic boundary conditions, and we introduce some 'defects' by changing the coupling constants at certain points $i = k_1, \ldots, k_M$ (with $M \ll N$) of the lattice.

We have considered the following class of defects:

$$\alpha_i = \gamma_1 \qquad \beta_i = \gamma_2 \qquad \text{but} \quad \alpha_i = \gamma_2 \qquad \beta_i = \gamma_1 \qquad \text{if} \quad i = k_j\,. \tag{7}$$

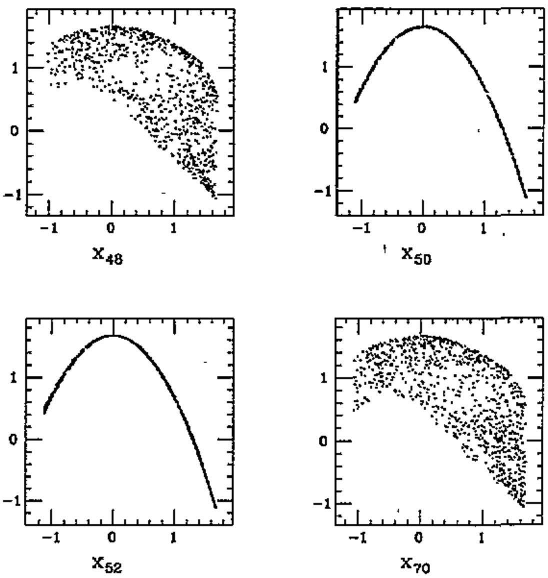

Figure 1. $x_i(n+1)$ versus $x_i(n)$ for $i = 48$, 50, 52, 70 when $N = 100$ and the defect is localized at $i = 50$. $\gamma_1 = 0.7$, $\gamma_2 = 0.01$.

The dynamic evolution of a defect, i.e. of a variable with the exchanged couplings, x_{k_j}, is essentially given by the single map solution $\widetilde{x}$, equation (5), as can be seen in a return map, see figure 1. Thus the motion of the defects are not very much influenced by the other variables. Yet there is a small synchronization length, l_c, of the variables near the defects: $x_i(n) \simeq x_{k_j}(n)$ for $i - k_j \leqslant l_c$. Unlike the case discussed above, this synchronization length does not depend on the noise in the numerical calculation. The doubling of the precision in the computation does not produce any change in l_c.

In spite of this apparently simple behaviour, the dynamics is very interesting if one studies the evolution in the tangent space. After a short transient, independent of both $\boldsymbol{x}(0)$ and $\boldsymbol{z}(0)$, the tangent vector $\boldsymbol{z}(n)$ becomes localized around one of the defects. This means that its components are sensibly different from zero only in a short region, l_0, about the defect. This region turns out to be of the order of the synchronization length, l_c.

The defect around which the tangent vector is localized changes in time. The passage from one defect to another happens suddenly, in few time iterations. The entire process can be visualized in terms of random jumping from one defect to another. In the meantime the modulus of the tangent vector grows exponentially revealing that the motion is chaotic.

One can generalize the analysis by considering the full set of Lyapunov vectors $z^{(j)}(n)$, with $j = 1, \ldots, N$, associated with the Lyapunov exponents λ_j. Each vector $z^{(j)}(n)$ evolves according to (6), but with orthogonal initial conditions, i.e. $z^{(j)}(0) \perp z^{(j')}(0)$ if $j \neq j'$. The Lyapunov exponents measure the exponential growth of the volume individuated by the Lyapunov vectors. More explicitly

$$\lambda_1 + \lambda_2 + \cdots + \lambda_q = \lim_{n\to\infty} \frac{1}{n} \ln \frac{|z^{(1)}(n) \wedge z^{(2)}(n) \wedge \cdots \wedge z^{(q)}(n)|}{|z^{(1)}(0) \wedge z^{(2)}(0) \wedge \cdots \wedge z^{(q)}(0)|} \tag{8}$$

where $\wedge$ indicates the external product.

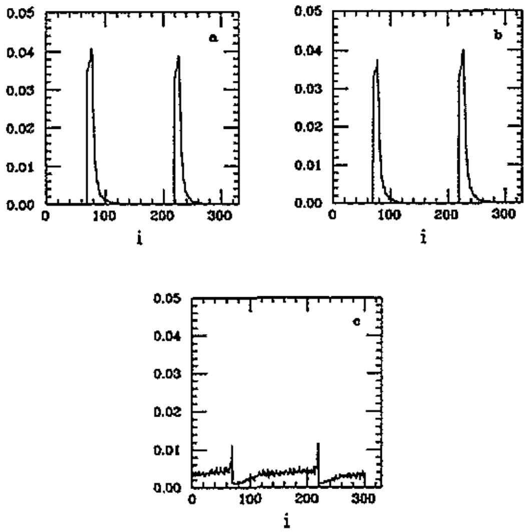

Figure 2. $\langle |\tilde{z}_i^{(j)}|^2 \rangle$ versus i for (*a*) $j = 1$, (*b*) $j = 2$ and (*c*) $j = 3$; $N = 300$ and two defects are present at $i = 70$ and $i = 220$. The evolution time is $T = 50\,000$.

The study of the time evolution of the tangent vectors for the system (1), (7) reveals that $z^{(j)}(n)$, for $j \leqslant M$ has the same qualitative behaviour of $z^{(1)}(n)$, while for $j > M$ does not localize. Figure 2 shows $\langle |\tilde{z}_i^{(j)}|^2 \rangle$ as a function of i, where

$$\langle |\tilde{z}_i^{(j)}|^2 \rangle = \lim_{T\to\infty} \frac{1}{T} \sum_{n=0}^{T-1} \frac{|z_i^{(j)}(n)|^2}{\sum_{l=1}^{N} |z_l^{(j)}(n)|^2} . \tag{9}$$

A quantitative characterization of the degree of localization may be obtained from the entropy [6]

$$H^{(j)} = -\sum_{i=1}^{N} \langle |\tilde{z}_i^{(j)}|^2 \rangle \ln \langle |\tilde{z}_i^{(j)}|^2 \rangle . \tag{10}$$

We find that $H^{(j)}$, when $j \leqslant M$, depends on the density of the defects, $\rho = M/N$, and that there is a critical density $\rho_c \simeq l_c/N$. For $\rho < \rho_c$ we have:

$$H^{(j)} - \ln N \sim c \ln \rho \tag{11}$$

where c depends on γ_1 and γ_2, while for $\rho > \rho_c$ all the entropies H^j saturate to a constant value, see figure 3.

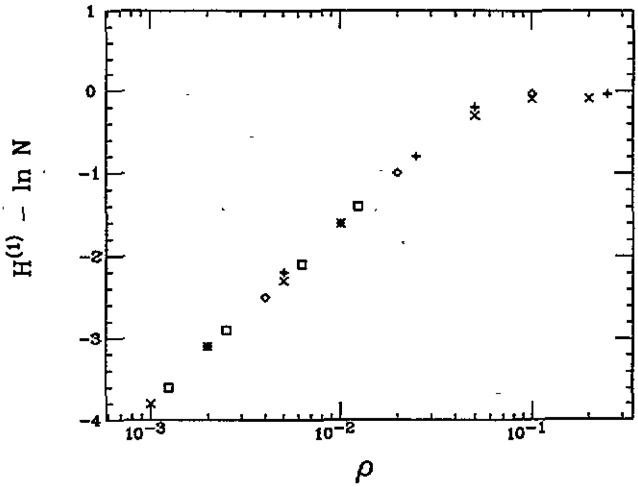

Figure 3. $H^{(1)} - \ln N$ versus ρ for $N = 200, 300, 500, 800, 1000$.

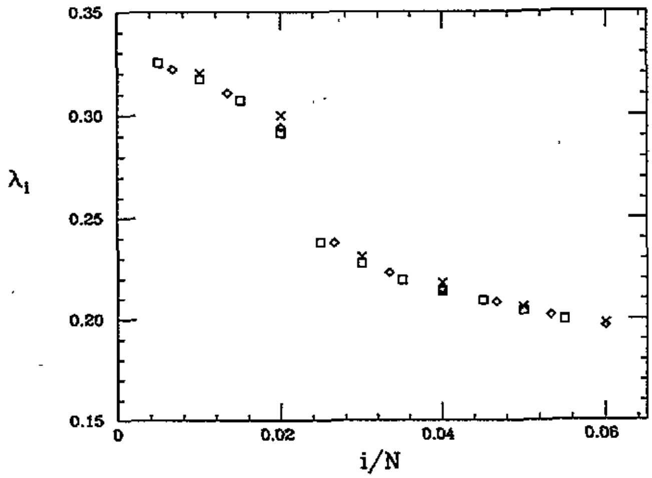

Figure 4. Higher positive Lyapunov exponents, λ_i, versus i/N for $N =$ 100 (crosses), $N = 150$ (diamonds) and $N = 200$ (squares), with a constant density of defects $\rho = 1/50$.

Finally we note that the Lyapunov exponents tend to cluster according to the number of the defects, so that the spectrum λ_i depends only on the density of the defects, and exhibits the following behaviour

$$\lambda_i \simeq \lambda_1 h_\rho (i/N) \tag{12}$$

where λ_1 depends only on the parameter of the local map and $h_\rho(i/N)$ is a ρ-dependent function of i/N, see figure 4. Formula (12) is well known [7] in the case of coupled maps without asymmetric coupling and in the limit of large N.

Also these results do not depend on the noise level in the numerical calculations.

From our analysis, by means of Lyapunov exponents and tangent vectors, we have the following scenario. The system is driven by the defects, which practically are not influenced by the rest of the system; moreover chaos is concentrated on these defects, and the density of the defects is the relevant property to determine the features of the Lyapunov exponents. Something similar to a deconfining transition concerning the localization properties of chaos appears only for density of defects large enough to permit considerable overlaps between the coherent regions of two adjacent defects.

We conclude by noting that the main features of this scenario also hold for other classes of defects, e.g.

$$\alpha_i = \gamma_1 \qquad \beta_i = \gamma_2 \qquad \text{but} \quad \alpha_i = \beta_i = 0 \qquad \text{if} \quad i = k_j \,.$$

This means that as long as the chain is not broken (like in the case of Aranson *et al*) our results are quite robust and everything is independent of numerical precision.

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